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# ON THE NON-ANALYTICITY LOCUS OF AN ARC-ANALYTIC FUNCTION

KRZYSZTOF KURDYKA AND ADAM PARUSIŃSKI

**ABSTRACT.** A function is called arc-analytic if it is real analytic on each real analytic arc. In real analytic geometry there are many examples of arc-analytic functions that are not real analytic. Arc analytic functions appear while studying the arc-symmetric sets and the blow-analytic equivalence. In this paper we show that the non-analyticity locus of an arc-analytic function is arc-symmetric. We discuss also the behavior of the non-analyticity locus under blowings-up. By a result of Bierstone and Milman a big class of arc-analytic function, namely those that satisfy a polynomial equation with real analytic coefficients, can be made analytic by a sequence of global blowings-up with smooth centers. We show that these centers can be chosen, at each stage of the resolution, inside the non-analyticity locus.

## 1. INTRODUCTION.

Let  $X$  be a real analytic manifold. A function  $f : X \rightarrow \mathbb{R}$  is called *arc-analytic*, cf. [12], if for every real analytic  $\gamma : (-1, 1) \rightarrow X$  the composition  $f \circ \gamma$  is analytic. The arc-analytic functions are closely related to blow-analytic functions of Kuo, cf. [10]. In particular, we have the following result, conjectured for the functions with semi-algebraic graphs in [12], and shown in [2].

**Theorem 1.1.** *Let  $X$  be a nonsingular real analytic manifold and let  $f : X \rightarrow \mathbb{R}$  be an arc-analytic function on  $X$ . Suppose that*

$$G(x, f(x)) = 0,$$

where

$$G(x, y) = \sum_{i=0}^p g_i(x) y^{p-i}$$

*is a nonzero polynomial in  $y$  with coefficients  $g_i(x)$  which are analytic functions on  $X$ . Then there is a mapping  $\pi : X' \rightarrow X$  which is a composite of a locally finite sequence of blowings-up with nonsingular closed centers, such that  $f \circ \pi$  is analytic.*

Let  $f : X \rightarrow \mathbb{R}$  be an arc-analytic subanalytic function. In this paper we study the set  $S(f)$  of non-analyticity of  $f$ . By definition,  $S(f)$  is the complement of the set  $R(f)$  of points  $p \in X$ , such that  $f$  as a germ is real analytic at  $p$ . It is known (cf. [17], [11], [1]) that  $S(f)$  is closed and subanalytic. It follows from [2] or [16], that  $\dim S(f) \leq \dim X - 2$ . As we show in Theorem 3.1 below,  $S(f)$  is arc-symmetric in the sense of [12]. Theorem 3.1 is shown in section 3.

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We also study how the set of non-analyticity behaves under blowings-up with smooth centers. This depends on whether the center is entirely contained in  $S(f)$  or not. If it is not then the non-analyticity lifts to the entire fiber, see Proposition 3.10. Note that Theorem 1.1 can be also derived from [16]. Using the method of [16] and Proposition 3.10 we show the following refinement of Theorem 1.1.

**Theorem 1.2.** *In Theorem 1.1 we may require that the mapping  $\pi : X' \rightarrow X$ , that is a locally finite composite  $\pi = \cdots \circ \pi_k \circ \cdots \circ \pi_0$  of blowings-up with smooth centers, satisfies additionally:*

*for every  $k$  the center of  $\pi_{k+1}$  is contained in the locus of non-analyticity of  $f \circ \pi_0 \circ \cdots \circ \pi_k$ .*

**1.1. Algebraic case.** Theorem 1.1 can be stated in the real algebraic version, see [2]. In this case if we assume that  $X$  is a nonsingular real algebraic variety and that the coefficients  $g_i$  are regular then we may require that  $\pi$  is a finite composite of blowings-up with nonsingular algebraic centers.

In the algebraic case we cannot require that the centers of blowings-up are entirely contained in the non-analyticity loci as Example 1.5 shows.

An analytic function on  $X$  is called *Nash* if its graph is semialgebraic. It is called *blow-Nash* if it can be made Nash after composing with a finite sequence of blowing-ups with smooth nowhere dense regular centers. Thus the algebraic version of Theorem 1.1, cf. [2], says that the function with semi-algebraic graph is arc-analytic if and only if it is blow-Nash. Nash morphisms and manifolds form a natural category that contains the algebraic one, cf. [4]. We note that our refinement of the statement of Theorem 1.1 holds in the Nash category.

**Theorem 1.3.** *Let  $X$  be a Nash manifold and let  $f : X \rightarrow \mathbb{R}$  be an arc-analytic function on  $X$ . Suppose that*

$$G(x, f(x)) = 0,$$

*where*

$$G(x, y) = \sum_{i=0}^p g_i(x) y^{p-i}$$

*is a nonzero polynomial in  $y$  with coefficients  $g_i(x)$  which are Nash functions on  $X$ . Then there is a finite composite  $\pi = \cdots \circ \pi_k \circ \cdots \circ \pi_0$  of blowings-up of nonsingular Nash submanifolds, such that for every  $k$  the center of  $\pi_{k+1}$  is contained in the locus of non-analyticity of  $f \circ \pi_0 \circ \cdots \circ \pi_k$ , and  $f \circ \pi$  is Nash.*

**1.2. Subanalytic case.** Less is known for an arc-analytic function with subanalytic graph if it does not satisfy an equation (1.1). It is known that an arc-analytic subanalytic function has to be continuous and can be made real analytic by composing with finitely many local blowings-up with smooth centers, see [2] or [16] (we refer the reader to these papers for a precise statement). It is not known whether these blowings-up can be made global that is whether the arc-analytic subanalytic functions coincide with the family of blow-analytic functions of T.-C.Kuo, see e.g. [10], [6], [7]. It is also not known, whether the centers of such blowings-up can be chosen in the locus of non-analyticity of the function.

We present below in Example 1.6 a subanalytic arc-analytic function that cannot be made analytic, even locally, by a blowing-up of a coherent ideal. In particular, it cannot satisfy an equation of type (1.1).

### 1.3. Examples.

*Examples 1.4.* The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x, y) = \frac{x^3}{x^2+y^2}$  for  $(x, y) \neq (0, 0)$  and  $f(0, 0) = 0$ , is arc-analytic but not differentiable at the origin.

The function  $g(x, y) = \sqrt{x^4 + y^4}$  is arc-analytic but not  $C^2$ . This example is due to E. Bierstone and P.D. Milman.

The function  $h : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $h(x, y) = \frac{xy^5}{x^4+y^6}$  for  $(x, y) \neq (0, 0)$  and  $h(0, 0) = 0$  is arc-analytic but not lipschitz. This example is due to L. Paunescu.

We generalize the first example as follows. Fix a real analytic Riemannian metric on  $X$  and let  $Y$  be a nonsingular real analytic subset of  $X$ . Then  $d_Y^2 : X \rightarrow \mathbb{R}$ , the square of the distance to  $Y$ , is a real analytic function on  $X$ . Suppose that  $Y$  is of codimension  $\geq 2$  in  $X$  and let  $f : X \rightarrow \mathbb{R}$  be an analytic function vanishing on  $Y$  and not divisible by  $d^2$ . Then,  $\frac{f^3}{d^2}$  vanishes on  $Y$ , is arc-analytic and not analytic at the points of  $Y$ . Note that  $\frac{f^3}{d^2}$  composed with the blowing-up of  $Y$  is analytic.

*Example 1.5.* Let  $g(x, y) = y^2 + x(x-1)(x-2)(x-3)$ . Then  $g^{-1}(0) \subset \mathbb{R}^2$  is irreducible and has two connected compact components, denoted by  $X_1$  and  $X_2$ . These connected components that can be separated by  $h(x, y) = x - 1.5$ , that is  $h < 0$  on  $X_1$  and  $h > 0$  on  $X_2$ . For  $\varepsilon > 0$  sufficiently small,  $h^2 + \varepsilon g$  is strictly positive on  $\mathbb{R}^2$ . Define

$$g_1(x, y) = \sqrt{h^2 + \varepsilon g} + h.$$

Then  $g_1$  is analytic, 0 is a regular value of  $g_1$  and  $g_1^{-1}(0) = X_1$ . Moreover,  $g_1$  is Nash. Then  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  defined by

$$f(x, y, z) = \frac{z^3}{z^2 + g_1^2(x, y)}$$

for  $(x, y, z) \neq 0$  and  $f(0) = 0$ , is arc-analytic and  $S(f) = X_1 \times \{0\}$ . The function  $f$  becomes analytic after blowing-up of  $S(f)$ .

*Example 1.6.* Let  $\pi_0 : \tilde{\mathbb{R}}^3 \rightarrow \mathbb{R}^3$  be the blowing-up of the origin and let  $E$  be the exceptional divisor of  $\pi_0$ . Let  $C \subset E$  be a transcendental (the smallest algebraic subset of  $E$  that contains  $C$  is  $E$  itself) non-singular analytic curve and let  $\pi_C : M \rightarrow \tilde{\mathbb{R}}^3$  be the blowing-up of  $C$ . Let  $f$  be an arc-analytic function on  $\mathbb{R}^3$  such that the set of non-analyticity of  $f \circ \pi_0$  is  $C$  and  $f \circ \pi_0 \circ \pi_C$  is analytic. Such a function can be constructed as follows. Using the last remark of Examples 1.4 we may construct an arc-analytic function  $g : \tilde{\mathbb{R}}^3 \rightarrow \mathbb{R}$  such that  $S(g) = C$ . Then we may set  $f(x, y, z) = (x^2 + y^2 + z^2) g(\pi_0^{-1}(x, y, z))$ .

Such  $f$ , as a germ at 0, cannot be made analytic by a single blowing-up of an ideal. Indeed, suppose contrary to our claim that there exists an ideal  $\mathcal{I}$  of  $\mathbb{R}\{x_1, x_2, x_3\}$  such that  $f \circ \pi_{\mathcal{I}}$  is analytic, where  $\pi_{\mathcal{I}}$  denotes the blowing-up of  $\mathcal{I}$ . Multiplying  $\mathcal{I}$  by the maximal ideal at 0 we may assume that  $\pi_{\mathcal{I}}$  factors through  $\pi_0$ , i.e.  $\pi_{\mathcal{I}} = \pi_{\mathcal{J}} \circ \pi_0$ , where  $\mathcal{J}$  is a sheaf of coherent ideals centered on an algebraic subset  $Y$  of  $E$ . We may assume that  $\dim Y \leq 1$ . Thus the blowing-up of  $\mathcal{J}$ ,  $\pi_{\mathcal{J}} : M_{\mathcal{J}} \rightarrow \tilde{\mathbb{R}}^3$  is an isomorphism over the complement of  $Y$  that contradicts the construction of  $f$ .

## 2. ARC-MEROMORPHIC MAPPINGS.

In this section *subanalytic* mean subanalytic at infinity. Let us recall, [17], [11], that a subset  $A$  of  $\mathbb{R}^n$  is called *subanalytic at infinity* if  $A$  is subanalytic in some algebraic compactification of  $\mathbb{R}^n$ . (Then in fact it is subanalytic in every algebraic compactification of  $\mathbb{R}^n$ .) All functions and mappings are supposed to be subanalytic, that is their graphs are subanalytic at infinity.

**Definition 2.1.** Let  $U$  be an open subanalytic subset of  $\mathbb{R}^n$ . An everywhere defined subanalytic mapping  $f : U \rightarrow \mathbb{R}^m$  is called *arc-meromorphic* if for any analytic arc  $\gamma : (-1, 1) \rightarrow U$  there exists a discrete set  $D \subset (-1, 1)$  and  $\varphi$  an meromorphic function on  $(-1, 1)$  with poles contained in  $D$  and such that  $f \circ \gamma = \varphi$  on  $(-1, 1) \setminus D$ . Note that it may happen that  $f \circ \gamma$  does not coincide with  $\varphi$  at some points of  $D$  and may be at these points discontinuous.

*Example 2.2.* The function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by  $f(x, y) = \frac{xy}{x^2+y^2}$  for  $(x, y) \neq (0, 0)$  can be extended to an arc-meromorphic function on  $\mathbb{R}^2$  by assigning any value at the origin. Then it becomes discontinuous at  $(0, 0)$  even if for every analytic arc  $\gamma : (-1, 1) \rightarrow \mathbb{R}^2$ ,  $\gamma(0) = (0, 0)$ ,  $f \circ \gamma$  extends to an analytic function.

*Remark 2.3.* If  $f$  is an arc-meromorphic and continuous function on an open connected set  $U \subset \mathbb{R}^n$ , then  $f$  is arc-analytic.

*Remark 2.4.* Let  $f$  and  $g$  be arc-meromorphic functions on an open connected set of  $U$ . Assume that  $f = g$  on an open non-empty subset  $U \subset \mathbb{R}^n$ , then  $f = g$  except on a nowhere dense subanalytic subset of  $U$ .

**Lemma 2.5.** *Let  $U$  be an open bounded subanalytic subset in  $\mathbb{R}^n$  and  $f : U \rightarrow \mathbb{R}^m$  be an arc-meromorphic mapping. Then there exists  $\Gamma \subset \mathbb{R}^n$  a closed nowhere dense subanalytic set,  $N \in \mathbb{N}$  and  $C > 0$  such that*

$$(1) \quad |f(x)| \leq C \operatorname{dist}(x, \Gamma)^{-N}, \quad x \in U \setminus \Gamma.$$

*In particular we can take as  $U$  the complement of the non-analyticity locus of  $f$ .*

*Proof.* It is well-known (cf. e.g. [9], [15]) that there exists a stratification of  $\mathbb{R}^n$  which is compatible with  $\overline{U}$  and such that  $f$  is analytic on each stratum contained in  $U$ . We take as  $\Gamma$  the union of all strata contained in  $\overline{U}$  of dimension less than  $n$ . Let us consider the function defined as follows:  $g(x) = |f(x)|$  if  $|f(x)| \leq 1$ , and  $g(x) = |f(x)|^{-1}$  if  $|f(x)| \geq 1$ . Then  $h(x) := \operatorname{dist}(x, \Gamma)g(x)$  is a subanalytic and continuous function on  $\overline{U}$  which is compact. Moreover, if  $\operatorname{dist}(x, \Gamma) = 0$  then  $h(x) = 0$ . Therefore, by the classical Łojasiewicz's inequality (cf. e.g. [9], [1]) for subanalytic functions, there exist  $N \in \mathbb{N}$  and  $c > 0$  such that

$$(2) \quad h(x) \geq c \operatorname{dist}(x, \Gamma)^{N+1}, \quad x \in U.$$

Thus inequality (1) follows with  $C = \max\{1/c, M\}$ , where  $M = \sup_{x \in U} \operatorname{dist}(x, \Gamma)^N$ .  $\square$

We state now an auxiliary lemma on arc-meromorphic functions in two variables.

**Lemma 2.6.** *Let  $U$  be an open subanalytic subset in  $\mathbb{R}^2$  and let  $f : U \rightarrow \mathbb{R}^m$  be an arc-meromorphic mapping. Then for any  $a \in U$  there exists a neighborhood  $V$  of  $a$  and an analytic function  $\varphi : V \rightarrow \mathbb{R}$ ,  $\varphi \not\equiv 0$ , such that  $\varphi f$  is arc-analytic.*

*Proof.* Let  $\Gamma$  be the subanalytic set associated to  $f$  by Lemma 2.5. Clearly we may assume that  $a \in \Gamma$ , otherwise  $f$  is analytic at  $a$  and the statement is trivial. Since  $\dim \Gamma = 1$ , by a result of Lojasiewicz's [14] (see also [13]), the set  $\Gamma$  is actually semianalytic. Then there exists a neighborhood  $V$  of  $a$  and an analytic function  $\psi : V' \rightarrow \mathbb{R}$ ,  $\psi \not\equiv 0$ , which vanishes on  $V' \cap \Gamma$ . Hence for some compact neighborhood  $V \subset V'$  of  $a$  there exists  $c > 0$  such that

$$|\psi(x)| \leq c \operatorname{dist}(x, \Gamma), \quad x \in V.$$

(This is a consequence of the main value theorem). Put  $\varphi = \psi^{N+1}$ , then by Lemma 2.5 the function  $\varphi f$  is continuous on  $V$ . Clearly  $\varphi f$  is arc-meromorphic, so by Remark 2.3 this function is arc-analytic.  $\square$

**Proposition 2.7.** *Let  $f : U \rightarrow \mathbb{R}$  be an arc-meromorphic function, where  $U$  is an open subset in  $\mathbb{R}^n$ . Assume that  $f$  is analytic with respect to the variable  $x_1$ . Then the function  $\frac{\partial f}{\partial x_1} : U \rightarrow \mathbb{R}$  is again arc-meromorphic.*

*Proof.* First observe that by [11] the function  $\frac{\partial f}{\partial x_1}$  is (globally) subanalytic. To prove that  $\frac{\partial f}{\partial x_1}$  is arc-meromorphic let us fix an analytic arc  $\gamma : (-1, 1) \rightarrow U$ . We define an arc-meromorphic function  $g : V \rightarrow \mathbb{R}$  by  $g(s, t) = f(\gamma(t) + se_1)$ , where  $e_1 = (1, 0, \dots, 0)$  and  $V$  is an open neighborhood of  $\{0\} \times (-1, 1)$  in  $\mathbb{R}^2$ . Clearly

$$\frac{\partial f}{\partial x_1}(\gamma(t)) = \frac{\partial g}{\partial s}(0, t).$$

By Lemma 2.6 there exist a neighborhood  $V$  of  $(0, 0)$  and an analytic function  $\varphi : V \rightarrow \mathbb{R}$  such that  $h := \varphi g$  is arc-analytic on  $V$ . Since  $\dim S(h) \leq 0$ , for any  $t \neq 0$  sufficiently small  $h$  is analytic at  $(0, t)$ , but of course also  $\varphi$  is analytic at  $(0, t)$ . Since  $g(s)$  is analytic with respect to  $s$  it follows that  $g = h/\varphi$  is actually analytic at  $(0, t)$  for any  $t \neq 0$  sufficiently small. By [2] there exists a map  $\pi : M \rightarrow \mathbb{R}^2$ , which is a finite composition of blowing-ups of points, such that  $h \circ \pi$  is analytic. Consider the arc  $\eta(t) := (0, t)$  and let  $\tilde{\eta}(t) \in M$  be the unique analytic arc such that  $\pi \circ \tilde{\eta} = \eta$ . The chain rule gives

$$(3) \quad d_{\tilde{\eta}(t)} h \circ \pi = (d_{\eta(t)} h) \circ (d_{\tilde{\eta}(t)} \pi).$$

Note that  $d_{\tilde{\eta}(t)} \pi$  is invertible for  $t \neq 0$ , moreover the map  $t \mapsto (d_{\tilde{\eta}(t)} \pi)^{-1}$  is meromorphic. It follows that  $t \mapsto d_{\eta(t)} h$  is meromorphic. In particular  $t \mapsto \frac{\partial h}{\partial s}(0, t)$  is meromorphic. We have

$$\frac{\partial h}{\partial s}(0, t) = \varphi \frac{\partial g}{\partial s}(0, t) + g \frac{\partial \varphi}{\partial s}(0, t).$$

Since  $\varphi(0, t) \neq 0$  for  $t \neq 0$ , the map  $t \mapsto \frac{\partial g}{\partial s}(0, t)$  is meromorphic and Proposition 2.7 follows.  $\square$

*Remark 2.8.* A repeated application of Proposition 2.7 shows that for every  $k \in \mathbb{N}$ ,

$$\frac{\partial^k f}{\partial x_1^k} : U \rightarrow \mathbb{R}$$

is arc-meromorphic. Moreover, there exists a subanalytic stratification  $\mathcal{S}$  of  $U$  such that for every stratum  $S \in \mathcal{S}$  and every  $x \in S$  there is  $\varepsilon > 0$  and a neighborhood  $V$  of  $x$  in  $S$  such that  $f(x + se_1)$  is an analytic function of  $(x, s) \in V \times (-\varepsilon, \varepsilon)$ . In particular, for every  $k \in \mathbb{N}$ ,  $\partial^k f / \partial x_1^k : U \rightarrow \mathbb{R}$  is analytic on the strata of  $\mathcal{S}$ .

### 3. THE NON-ANALYTICITY LOCUS OF AN ARC-ANALYTIC FUNCTION IS ARC-SYMMETRIC.

Let  $U \subset \mathbb{R}^n$  be open and let  $f : U \rightarrow \mathbb{R}$  be arc-analytic with subanalytic graph. We denote by  $S(f)$  the non-analyticity set of  $f$  and by  $R(f)$  its complement in  $U$ . Then  $S(f)$  is closed in  $U$  and by [17] (see also [11], [2]) it is a subanalytic set. It follows from [2] or [16] that  $\dim S(f) \leq n - 2$ .

**Theorem 3.1.** *Let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow U$  be an analytic arc such that  $\gamma(t) \in R(f)$  for  $t < 0$ . Then  $\gamma(t) \in R(f)$  for  $t > 0$  and small. In other words,  $S(f)$  is arc-symmetric subanalytic in the sense of [12].*

For the proof we need some basic properties of Gateaux differentials. For each  $k \in \mathbb{N}$  we consider

$$(4) \quad h_k(x, v) = \frac{1}{k!} \partial_v^k f(x) = \frac{1}{k!} \frac{d^k}{dt^k} f(x + tv)|_{t=0}.$$

**Proposition 3.2.** *Let  $f : U \rightarrow \mathbb{R}$  be an arc-analytic function. Then for any  $k \in \mathbb{N}$  the function  $h_k(x, v) : U \times \mathbb{R}^n \rightarrow \mathbb{R}$  is arc-meromorphic.*

*Proof.* Let  $(x(t), v(t))$  be an analytic arc in  $U \times \mathbb{R}^n$ . Define an arc-analytic function  $g(s, t) = f(x(t) + sv(t))$ . Then

$$h_k(x(t), v(t)) = \frac{1}{k!} \frac{\partial^k}{\partial s^k} g(t, s)|_{s=0}$$

that is meromorphic by Proposition 2.7. □

For  $x \in U$ ,  $k \in \mathbb{N}$  we denote

$$h_{x,k}(v) = h_k(x, v) = \frac{1}{k!} \partial_v^k f(x)$$

Note that  $h_{x,k}$  is  $k$ -homogeneous function. If  $f$  is analytic at  $x$ , then  $h_{x,k}$  is polynomial. We have also the inverse which is Bochnak-Siciak Theorem, see [5], which states that if  $h_{x,k}$  is polynomial for each  $k \in \mathbb{N}$ , then  $f$  is analytic at  $x$ . Traditionally if  $h_{x,k}$  is polynomial then it is called the Gateaux differential of  $f$  at  $x$  of order  $k$ .

We call  $h_{x,k}$  *generically polynomial* if it is equal to a polynomial except on a nowhere dense subanalytic (and homogenous) subset of  $\mathbb{R}^n$ . Note that, by Remark 2.4,  $h_{x,k}$  is generically polynomial if it coincides with a polynomial on an open nonempty set.

**Proposition 3.3.** *Let  $f : U \rightarrow \mathbb{R}$  be an arc-analytic function, where  $U$  is an open subset in  $\mathbb{R}^n$ . Let  $\gamma : (-\varepsilon, \varepsilon) \rightarrow U$  be an analytic arc and  $k \in \mathbb{N}$ . If  $h_{\gamma(t),k}$  is generically polynomial for  $t \in (-\varepsilon, 0)$ , then there exists a finite set  $F_k \subset (0, \varepsilon)$  such that  $h_{\gamma(t),k}$  is generically polynomial for each  $t \in (0, \varepsilon) \setminus F_k$ .*

*Proof.* Let  $\mathbb{R}_k[x_1, \dots, x_n]$  denote the space of homogenous polynomials of degree  $k$  and let  $d_k = \binom{n+k-1}{n}$  denote its dimension. We need the classical multivariate interpolation.

**Lemma 3.4.** *There exists an algebraic nowhere dense subset  $\Delta \subset (\mathbb{R}^n)^{d_k}$  such that for  $V = (v^1, \dots, v^{d_k}) \in (\mathbb{R}^n)^{d_k} \setminus \Delta$  the map  $\Psi_V : \mathbb{R}_k[x_1, \dots, x_n] \rightarrow \mathbb{R}^{d(k)}$  given by*

$$\Psi_V(P) = (P(v^1), \dots, P(v^{d_k})).$$

*is a linear isomorphism.* □

Fix  $V = (v^1, \dots, v^{d_k}) \in (\mathbb{R}^n)^{d_k} \setminus \Delta$  generic and denote  $\Phi_V = \Psi_V^{-1} : \mathbb{R}^{d(k)} \rightarrow \mathbb{R}_k[x_1, \dots, x_n]$ . We define an arc-meromorphic map  $P_k : (-\varepsilon, \varepsilon) \rightarrow \mathbb{R}_k[x_1, \dots, x_n]$  by

$$P_k(t) := \Phi_V(h_k(\gamma(t), v^1), \dots, h_k(\gamma(t), v^{d(k)})).$$

The map  $p_k; (-\varepsilon, \varepsilon) \times \mathbb{R}^n \rightarrow \mathbb{R}$ , where  $p_k(t, v) = P_k(t)(v)$  is arc-meromorphic. If  $V$  is sufficiently generic then, for  $t \in (-\varepsilon, 0) \setminus \{\text{finite set}\}$ ,  $p_k(t)$  coincides with  $h_{\gamma(t), k}$ . Since they both are arc-meromorphic, by Remark 2.4 they coincide on  $(-\varepsilon, \varepsilon) \times \mathbb{R}^n \setminus Z_k$ , where  $Z_k$  is a closed subanalytic set with  $\dim Z_k \leq n$ . Hence there exists a finite set  $F_k \subset (0, \varepsilon)$  such that for  $t \in (0, \varepsilon) \setminus F_k$  the intersection  $Z_k \cap (\{t\} \times \mathbb{R}^n)$  is of dimension less than  $n$ . Thus, for each  $t \in (0, \varepsilon) \setminus F_k$  the function  $h_{\gamma(t), k}$  is generically polynomial, as claimed.  $\square$

The following proposition is a version of the mentioned above Bochnak-Siciak Theorem, [5].

**Proposition 3.5.** *If for every  $k$  there is a nonempty open subset  $V_k \subset \mathbb{R}^n$  and a homogeneous polynomial  $P_k$  of degree  $k$  such that  $h_{x, k} \equiv P_k$  on  $V_k$ , then  $f$  is analytic at  $x$ .*

*Proof.* We first show that  $\sum_k P_k(v)$  is convergent in a neighborhood of  $0 \in \mathbb{R}^n$ .

We may assume that  $x$  is the origin. Let  $\pi_0$  be the blowing up of the origin,  $\pi_0(y, s) = (sy, s)$ ,  $s \in \mathbb{R}$ ,  $y \in \mathbb{R}^{n-1}$ , in a chart. The function  $\tilde{f}(y, s) := f(\pi(y, s))$ , defined in a neighborhood  $U'$  of the exceptional divisor  $E : s = 0$ , is arc-analytic. The set of non-analyticity of  $\tilde{f}$ , denoted by  $\tilde{S}$ , is closed subanalytic and of codimension at least 2. For  $y \notin \tilde{S}$ ,  $\tilde{f}$  is analytic in a neighborhood of  $(0, y)$  and, moreover, by analytic continuation,

$$(5) \quad h_{x, k}(v) = P_k(v) \quad \text{for } v = t(y, 1), t \in \mathbb{R}, y \notin \tilde{S}.$$

Fix  $A'$  an open non-empty subset of  $E$  such that the closure of  $A'$  does not intersect  $\tilde{S}$ . Let  $A \subset \mathbb{R}^n$  be the cone over  $A'$ . Then, by (5),  $\sum_k P_k(v)$  is convergent in any compact subset of  $A$ . The convergence in a neighborhood of  $0$  in  $\mathbb{R}^n$  follows from the following lemma.

**Lemma 3.6.** *Let  $V \subset \mathbb{R}^n$  be starlike with respect to the origin,  $a \in V$ , and suppose that*

$$|P_k(v)| \leq L \quad \text{on } V' = a + V.$$

*Then*

$$|P_k(v)| \leq L \quad \text{on } \frac{1}{2e}V.$$

*Proof.* Since  $P_k$  is homogeneous of degree  $k$

$$(6) \quad P_k(v) = \frac{1}{k!} \sum_{s=0}^{s=k} (-1)^{k-s} \binom{k}{s} P_k(a + sv).$$

Indeed, (6) can be shown recursively on  $k$  using Euler's formula as follows. First note (6) holds for  $a = 0$  and the derivative of the RHS of (6) with respect to  $a$  equals

$$(7) \quad 0 = \frac{1}{k!} \sum_{s=0}^{s=k} (-1)^{k-s} \binom{k}{s} Q(a + sv),$$



where  $Q(x) = \sum_{i=1}^n a_i \frac{\partial P_k}{\partial x_i}(x)$  is a homogeneous polynomial of degree  $k-1$ . By the inductive assumption

$$\begin{aligned} \sum_{s=0}^{s=k} (-1)^{k-s} \binom{k}{s} Q(a+sv) &= \sum_{s=0}^{s=k-1} (-1)^{k-1-s} \binom{k-1}{s} Q(a+sv) + \\ &+ \sum_{s=1}^{s=k} (-1)^{k-s} \binom{k-1}{s-1} Q(a+sv) = -Q(v) + Q(v) = 0 \end{aligned}$$

This shows (6). Thus, if  $v \in \frac{1}{k}V$ ,  $|P_k(v)| \leq \frac{1}{k!}L \sum_{s=0}^k \binom{k}{s} = L \frac{2^k}{k!}$ , that means that for  $v \in \frac{1}{2e}V$

$$|P_k(v)| \leq L \frac{(2k)^k}{k!} \frac{1}{(2e)^k} \leq L.$$

This ends the proof of lemma 3.6.  $\square$

Then  $\sum_k P_k(v)$  is an analytic function in a neighborhood of the origin that coincides with  $f$  on a set with non-empty interior. Hence  $f(v) = \sum_k P_k(v)$  in a neighborhood of the origin. This shows proposition 3.5.  $\square$

*Proof of theorem 3.1.* We may assume that  $\gamma$  is injective otherwise the image of  $t > 0$  equals the image of  $t < 0$  and the statement is obvious. Let  $F := \bigcup F_k$ , where  $F_k$  are finite subsets of  $(0, \varepsilon)$  given by Proposition 3.3. Clearly the complement of  $F$  is dense in  $(0, \varepsilon)$ , so by Proposition 3.5 our function  $f$  is analytic at  $\gamma(t)$  for  $t \in G$ , where  $G$  is an open dense subset of  $(0, \varepsilon)$ . Hence theorem 3.1 follows.  $\square$

Consider the subanalytic sets

$$\begin{aligned} \tilde{R}_{k_0}(f) &= \{x \in U; \forall k \leq k_0, h_{x,k} \text{ is generically polynomial} \}, \\ R_{k_0}(f) &= \{x \in U; \forall k \leq k_0, h_{x,k} \text{ is polynomial} \}. \end{aligned}$$

Clearly  $\tilde{R}_{k+1}(f) \subset \tilde{R}_k(f)$  and  $R_{k+1}(f) \subset R_k(f)$ . We recall from [11] the following result

**Proposition 3.7.** [ [11], Proposition 4.4] *Let  $f : U \rightarrow \mathbb{R}$  be a subanalytic (not necessarily arc-analytic) function on an open bounded  $U \subset \mathbb{R}^n$ . Then for any compact  $K \subset U$  there is  $k \in \mathbb{N}$  such that  $R(f) \cap K = R_k(f) \cap K$ .*

**Proposition 3.8.** *For any compact  $K \subset U$  there is  $k \in \mathbb{N}$  such that  $R(f) \cap K = \tilde{R}_k(f) \cap K$ .*

*Proof.* By Remark 2.8 there exists a stratification  $\mathcal{S}$  of  $U \times S^{n-1}$  such that for every  $k$ ,  $h_k$  is analytic on the strata. Refining the stratification, if necessary, we may suppose that for every stratum  $S \subset U \times S^{n-1}$  its projection to  $U$  has all fibers of the same dimension. In the proof we use only these strata for which all the fibers of projection to  $U$  are of maximal dimension  $n-1$ . We denote the collection of them by  $\mathcal{S}_n$  and their union as  $Z$ . Now it is easy to adapt the proof of Lemma 6.1 of [11] (based on multivariate interpolation) and show the following lemma.

**Lemma 3.9.** *There are analytic subanalytic functions*

$$w_i : U \times S^{n-1} \rightarrow \mathbb{R}, \quad i \in \mathbb{N},$$

*analytic on each stratum of  $\mathcal{S}$  such that  $h_{x,i}$  is generically polynomial if and only if  $w_i \equiv 0$  generically on  $\{x\} \times S^{n-1}$ .*  $\square$

Now Proposition 3.8 follows from Lemma 2.5 of [11] that shows that for every stratum there exist  $k$  such that

$$\bigcap_{i=1}^{\infty} \{w_i = 0\} = \bigcap_{i=1}^k \{w_i = 0\}.$$

□

We complete this section with two results, one that controls the change of non-analyticity locus by blowings-up. This result will be crucial in the next section. The last result of this section, Proposition 3.11, though not used in this paper, indicates a possible analogy between our approach and the theory of complex analytic functions.

**Proposition 3.10.** *Let  $T = \{x_k = x_{k+1} = \dots = x_n = 0\}$  and let  $\pi_T$  be the blowing-up of  $T$ . Suppose that the origin is in the closure of  $R(f) \cap T$  and that  $f \circ \pi_T$  is analytic at least at one point of  $\pi_T^{-1}(0)$  (hence on a neighborhood of this point). Then  $f$  is analytic at 0.*

*Proof.* Let  $\Pi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be given by  $\Pi(x, t, v) = x + tv$  and let  $\Pi_T : T \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  be the restriction of  $\Pi$ . First we show that if  $f \circ \Pi_T$  is analytic at some points of  $\Pi_T^{-1}(0) \cap \{t = 0\}$  and 0 is in the closure of  $R(f) \cap T$  then  $f$  is analytic at 0. Indeed, suppose that  $A' \subset \mathbb{R}^n$  has non-empty interior and suppose that  $f \circ \Pi_T$  is analytic in a neighborhood  $\{0\} \times \{0\} \times A'$ . Let  $h_k(x, v)$ ,  $x \in T, v \in \mathbb{R}^n$ , be defined by (4). Then  $h_k$  is arc-meromorphic and analytic on  $A = U' \times A'$ , where  $U'$  is a small neighborhood of 0 in  $T$ . For each  $k$ , we define by Lemma 3.4,

$$(8) \quad P_k(x, v) = \Psi_V^{-1}(h_k(x, v^1), \dots, h_k(x, v^{d(k)}))(v),$$

where  $v^1, \dots, v^{d(k)} \in A'$  are generic. Each  $P_k$  is analytic on  $A$  and equals  $h_k$  for  $x \in R(f) \cap T$ . Therefore  $h_k(0, v) = P_k(0, v)$  for  $v \in A'$  and the claim follows from proposition 3.5.

Thus it remains to show that  $f \circ \Pi_T$  is analytic at some points of  $\Pi_T^{-1}(0) \cap \{t = 0\}$ . For this we factor  $\Pi_T$  restricted to  $\{v_n \neq 0\}$  through  $\pi_T$  and use the assumption on  $\pi_T$ . Write  $\pi_T$  in an affine chart  $\pi_T(\tilde{x}, y, s) = (\tilde{x}, sy, s)$ , where  $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_{k-1})$ ,  $y = (y_k, \dots, y_{n-1})$  and  $s \in \mathbb{R}$ . Then on these charts  $\Pi_T = \pi_T \circ \varphi$ , where

$$(\tilde{x}, y, s) = \varphi(x, t, v) = (x + tv', \frac{1}{v_n}v'', tv_n),$$

where  $v' = (v_1, \dots, v_{k-1})$ ,  $v'' = (v_k, \dots, v_{n-1})$ . Restricted to  $t = 0$ ,  $\varphi$  is a surjective projection  $(x, v) \rightarrow (x, \frac{1}{v_n}v'')$  onto  $s = 0$ . Hence  $R(f \circ \Pi_T) \cap \Pi_T^{-1}(0) \cap \{t = 0\} \supset \varphi^{-1}(R(f \circ \pi_T) \cap \pi_T^{-1}(0))$  is non-empty. □

**Proposition 3.11.** *Let  $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$  and suppose that for every  $x_1 > 0$  and small,  $f(x_1, x')$  is analytic at  $(x_1, 0)$  as a function of  $x'$ . Moreover, suppose that for  $x_1 > 0$  and small we have a uniform bound*

$$|h_k((x_1, 0), v')| \leq c^k, \quad \text{for } \|v'\| \leq \varepsilon, k \in \mathbb{N},$$

where  $v' = (v_2, \dots, v_n)$ . Then  $f$  is analytic at the origin.

*Proof.* The function  $h_k((x_1, 0), v')$  is arc-meromorphic as a function of  $x_1, v'$ . Moreover, since continuous arc-meromorphic functions of one variable are analytic, using polynomial interpolation lemma, Lemma 3.4, we may show that each  $h_k((x_1, 0), v')$  extends to an analytic function  $\Psi(x_1, v')$  defined in a neighborhood of  $(0, 0)$ , such that for each  $x, v' \rightarrow \Psi(x_1, v')$  is a homogeneous polynomial in  $v'$ . Moreover, for  $x_1 > 0$  and  $\|x'\| < \varepsilon/c$

$$f(x_1, x') = \sum_k h_k((x_1, 0), x')$$

and the series on the right-hand side is convergent.

Fix any  $k \in \mathbb{N}$  and  $\|v'\| < \varepsilon/c$ . Then for  $v = (1, v')$ ,  $0 < t < 1$ ,

$$f(tv) = \sum_{j=0}^{\infty} h_j((t, 0), tv') = \sum_{j=0}^{\infty} t^j h_j((t, 0), v') = \sum_{j=0}^k t^j h_j((t, 0), v') + \varphi(t, v'),$$

where  $\varphi$  is subanalytic and  $O(t^{k+1})$ . Therefore for such  $v$

$$(9) \quad H_k(0, v) := \frac{1}{k!} \frac{d^k}{dt^k} f(tv)|_{t=0} = \frac{1}{k!} \frac{d^k}{dt^k} \sum_{j=0}^k h_j((t, 0), tv')|_{t=0}.$$

Note that the right-hand side, and hence  $H_k(0, v)$  as well, is a polynomial in  $v$ . Indeed, this follows from the fact that  $x \rightarrow \sum_{j=0}^k h_j((x_1, 0), x')$  is an analytic function of  $x$  and  $H_k(0, v)$  coincides with its Gateaux differential. Thus proposition 3.11 follows from proposition 3.5.  $\square$

#### 4. PROOF OF THEOREM 1.2.

We may suppose that  $U$  is connected. We suppose also that the coefficients  $g_0$  and  $g_p$  of  $G$  and the discriminant  $\Delta(x)$  of  $G$  are not identically equal to zero. By the resolution of singularities [8], [3], [18], there is a locally finite sequence of blowings-up  $\pi : U' \rightarrow U$  with nonsingular centers such that  $(g_0 g_p \Delta) \circ \pi$  is normal crossings. Thus Theorem 1.1 follows from the following.

**Proposition 4.1.** *Let an arc-analytic function  $f(x)$  satisfy the equation (1.1) with analytic coefficients  $g_i$ . If  $g_0$ ,  $g_p$  and  $\Delta(x)$  are simultaneously normal crossings (and hence not identically equal to zero) then  $f$  is real analytic.*

Proposition 4.1 was proven in [16] under an additional assumption  $g_0 \equiv 1$ , see the proof of Theorem 3.1 of [16]. It is easy to reduce the proof to this case by replacing  $f$  by  $g_p f$ . Then, an argument of [16] shows that locally  $f$  can be expand as a fractional power series. Finally, an arc-analytic fractional power series is analytic, see the proof of Theorem 3.1 of [16]. If the discriminant of  $G$  vanishes identically then we replace it by the first non-vanishing higher order discriminant.

To show Theorem 1.2 we follow, for the product  $h(x) = g_0(x)g_p(x)\Delta(x)$ , the monomialisation procedure of Włodarczyk or Bierstone-Milman. In this procedure the centre of blowing-up is defined as a the locus of points where a local invariant is maximal. Thus suppose that we have the following data described in a local system of coordinates  $x_1, \dots, x_n$  at the origin. The function  $h \circ \pi$ , where  $\pi = \pi_k \circ \dots \circ \pi_0$ , is of the form  $h \circ \pi = x^A h_k$ , where  $h_k$  is the controlled transform by the preceding blowings-up. Let  $m = \text{ord}_x h_k$ . We may assume that  $H = \{x_n = 0\}$  is a hypersurface of maximal contact. Then, using the notation  $x = (x', x_n)$ ,

$$h_k(x) = x_k^m + \sum_{j=0}^{m-2} c_j(x') x_k^j,$$

and  $\text{mult}_0 c_i \geq m - i$ .

Let  $C$  be the next centre given by the procedure and denote by  $\pi_C$  the blowing-up of  $C$ . We show that it cannot happen that  $0 \in S(h \circ \pi)$  and  $0 \in \overline{C} \setminus S(f \circ \pi)$ . Suppose, contrary to our claim, that this is the case. Then, by Proposition 3.10, the fibre over the origin of the blowing-up  $\pi_C = \pi_{k+1}$  of  $C$  is contained in  $S(f \circ \pi \circ \pi_C)$ . Since  $C$  is contained in

the equimultiplicity locus of  $h_k$ , at the generic point  $\pi_C^{-1}(0)$  the strict transform of  $h_k$  is nonzero, and hence  $h \circ \pi \circ \pi_C$  is normal crossing. This contradicts Proposition 4.1.

Let  $C'$  denote the connected component of  $C$  containing 0. Then either  $C' \subset S(h \circ \pi)$  or  $C' \cap S(h \circ \pi) = \emptyset$ . Thus Theorem 1.2 proven.  $\square$

## REFERENCES

- [1] E. Bierstone and P. D. Milman, *Semianalytic and Subanalytic sets*, Publ. I.H.E.S., **67** (1988), 5-42.
- [2] E. Bierstone, P. D. Milman, *Arc-analytic functions*, Invent. math. **101** (1990), 411-424.
- [3] E. Bierstone, P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207-302.
- [4] J. Bochnak, M. Coste, M.-F. Roy, *Géométrie algébrique réelle*, E.M.G vol. 36 (1998) Springer.
- [5] J. Bochnak, J. Siciak, *Analytic functions in topological vector spaces*, Studia Math., **XXXIX** (1971), 77-112
- [6] T. Fukui, S. Koike, T.-C. Kuo, *Blow-analytic equisingularities, properties, problems and progress*, Real Analytic and Algebraic Singularities (T. Fukuda, T. Fukui, S. Izumiya and S. Koike, ed), Pitman Research Notes in Mathematics Series, **381** (1998), pp. 8-29.
- [7] T. Fukui, L. Paunescu, *On Blow-analytic Equivalence*, in "Arc Spaces and Additive Invariants in Real Algebraic and Analytic Geometry", Panoramas et Synthèses, S.M.F., **24**, 2007, 87-125
- [8] H. Hironaka, *Resolution of Singularities of an algebraic variety over a field of characteristic zero, I-II* Ann. of Math., **97** (1964).
- [9] H. Hironaka, *Subanalytic sets*, in Number Theory, Algebraic Geometry and Commutative Algebra (Kinokuniya, Tokyo), 1973, volume in honor of Yasuo Akizuki, 453-493.
- [10] T.-C. Kuo, *On classification of real singularities*, Invent. math. **82** (1985), 257-262.
- [11] K. Kurdyka, *Points réguliers d'un ensemble sous-analytique*, Ann. Inst. Fourier, Grenoble, **38** (1988), 133-156
- [12] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann. **281** no.3 (1988), 445-462
- [13] K. Kurdyka, S. Łojasiewicz, M. Zurro *Stratifications distinguées comme un outil en géométrie semi-analytique*, Manuscripta Math. **86**, 81-102, (1995).
- [14] S. Łojasiewicz, *Ensembles semi-analytiques*, preprint, I.H.E.S. (1965) available on the web page "http://perso.univ-rennes1.fr/michel.coste/Lojasiewicz.pdf".
- [15] S. Łojasiewicz, *Sur la géométrie semi- et sous-analytique*, Ann. Inst. Fourier (Grenoble) **43** (1993), 1575-1595.
- [16] A. Parusiński, *Subanalytic functions*, Trans. Amer. Math. Soc. **344**, 2 (1994), 583-595.
- [17] M. Tamm, *Subanalytic sets in the calculus of variation*, Acta Math. **146** (1981), no. 3-4, 167-199.
- [18] J. Włodarczyk, *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc., **18** (2005), no. 4, 779-822

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